

# Twisted Legendre transformation

S. Zakrzewski

Department of Mathematical Methods in Physics, University of Warsaw  
Hoża 74, 00-682 Warsaw, Poland

## Abstract

The general framework of Legendre transformation is extended to the case of symplectic groupoids, using an appropriate generalization of the notion of generating function (of a Lagrangian submanifold).

## 1 Tulczyjew triple and its generalization

The general framework of Legendre transformation was introduced by Tulczyjew [1]. It consists in recognition of the following structure, which we call the *Tulczyjew triple*:

$$T^*(TQ) \xleftarrow{\alpha} T(T^*Q) \xrightarrow{\beta} T^*(T^*Q). \quad (1)$$

Here  $Q$  is the manifold of *configurations* (of the system),  $T^*Q$  — its phase space (cotangent bundle),  $TQ$  — its velocity space (tangent bundle) and  $\alpha, \beta$  are natural symplectic isomorphisms. From those two isomorphisms,  $\beta$  is the easy one: it is just the vector bundle isomorphism induced by the symplectic form on  $P = T^*Q$  ( $\beta$  exists for any symplectic manifold  $P$ ). An explicit construction of  $\alpha$  can be found in [1, 2]. Let us mention here how the existence of such an isomorphism follows from the general theory of symplectic groupoids [3, 4]. Cotangent bundles are exactly those symplectic groupoids which have commutative multiplication and connected, simply connected fibers. The tangent bundle to such a symplectic groupoid is again a commutative symplectic groupoid with connected, simply connected fibers, hence a cotangent bundle to its ‘space of units’, which in this case coincides with  $TQ$ .

The dynamics is described by specifying a Lagrangian submanifold  $D \in T(T^*Q)$  (in the case of a section, we just have a Hamiltonian vector field on  $T^*Q$ ). The two isomorphisms  $\alpha$  and  $\beta$  allow to consider  $D$  as a Lagrangian submanifold of a cotangent bundle to  $TQ$  and to  $T^*Q$ , respectively. One can speak about generating functions of  $D$  in those two ‘control modes’. In the first case the function is called the *Lagrangian*, and in the second case it is (minus) the *Hamiltonian*. The passage from one formulation of dynamics to another consists in applying the *universal Legendre transformation* (1) to the particular case of Lagrangian submanifold.

The aim of this paper is to stress the fact that the above scheme can be generalized to the case when the cotangent bundle  $T^*Q$  is replaced by an arbitrary symplectic

groupoid. The set of units of the groupoid will be still denoted by  $Q$ . It is now a Poisson manifold. We shall restrict our attention to symplectic groupoids with connected and simply connected fibers (like cotangent bundles). They are known to be entirely determined by their ‘set of units’  $Q$  (with its Poisson structure) and therefore we denote such symplectic groupoids as  $\mathbf{Ph} Q$ . They provide a natural generalization of the phase space (or cotangent bundle) to the case of Poisson manifolds (those which are *integrable* in the sense that they are the set of units of a symplectic groupoid). We shall call  $\mathbf{Ph} Q$  the *phase space of  $Q$* , or, a *twisted cotangent bundle*. It is known [4] that any complete Poisson map of (integrable) Poisson manifolds is lifted to a unique morphism (in the sense of [4]) of the corresponding twisted cotangent bundles.

We can now state the generalization of (1), which may be called the *twisted Tulczyjew triple*:

$$\mathbf{Ph}(TQ) \xleftarrow{\alpha} T(\mathbf{Ph} Q) \xrightarrow{\beta} T^*(\mathbf{Ph} Q). \quad (2)$$

The isomorphism  $\beta$  is here the same as before and the isomorphism  $\alpha$  arises in a similar way as before: the tangent bundle to a symplectic groupoid over  $Q$  is a symplectic groupoid over  $TQ$ . One checks easily that the resulting Poisson structure on  $TQ$  is exactly the *tangent* Poisson structure [5] of the original structure on  $Q$ . Looking at (2) we see that the dynamics  $D \in T(\mathbf{Ph} Q)$  can be generated — in the usual sense — by a Hamiltonian, which is a function on the twisted cotangent bundle. In Section 3 we introduce the notion of a Lagrangian submanifold of a symplectic groupoid, *generated* by a function on its Poisson manifold of units. This will allow us to consider also the Lagrangian, which will be a function on  $TQ$  (on the usual tangent bundle, but generating in the twisted sense).

## 2 Some conventions

Recall that symplectic groupoids are symplectic-geometric models of involutive algebras (cf. *S\*-algebras* of [4]), the algebra multiplication map being replaced by a symplectic ‘multiplication’ relation (the groupoid multiplication). In particular, the cotangent bundle  $T^*Q$  (with its groupoid structure) is the symplectic counterpart of the algebra of functions on  $Q$ . On the other hand, from a *S\**-algebra we get a *S\**-coalgebra [4] just by inverting arrows and passing to the dual objects (for any symplectic manifold  $S$ , the role of its ‘dual’ is played by the symplectic manifold denoted by  $\overline{S}$  which is the same manifold considered with its opposite symplectic structure, i.e. minus the original one). In particular, the cotangent bundle  $T^*Q$  with its opposite symplectic structure and the ‘co-groupoid’ structure is the symplectic counterpart of the coalgebra of measures on  $Q$ . Note that we view  $\mathbf{Ph} Q$  rather as a coalgebra than an algebra, since in the first case the functorial correspondence between Poisson manifolds and their phase spaces is covariant.

Now we see that it becomes important to consider on any cotangent bundle both the canonical symplectic structure (the exterior derivative of the canonical Liouville 1-form) and its opposite one. As it is not clear which one should correspond to the cotangent bundle viewed as an ‘algebra’ (resp. ‘coalgebra’), we shall adopt our first

convention (it will be justified to some extent below, when considering the phase spaces of Poisson groups).

**Convention 1.** We shall denote by  $T^*Q$  the cotangent bundle to a manifold  $Q$  with the canonical symplectic structure and the groupoid (=‘algebra’) structure. By  $\mathbf{Ph} Q$  we shall denote the cotangent bundle with the opposite symplectic structure and the co-groupoid (=‘coalgebra’) structure.

The next convention concerns the choice of one of the two projections to define the correspondence between symplectic groupoids (or rather  $S^*$ -coalgebras) and Poisson manifolds.

**Convention 2.** The Poisson structure on  $Q$  coincides with the image of the Poisson structure on  $\mathbf{Ph} Q$  by the **left** groupoid projection.

The following convention may be viewed as a result of the two above conventions.

**Convention 3.** If  $G$  is a Lie group, then the tangent space at the group unit,  $\mathfrak{g} = T_e G$ , is equipped with the Lie bracket by identifying the elements of  $T_e G$  with the corresponding fundamental vector fields of the *left* translations on  $G$  (i.e. the right-invariant vector fields on  $G$ ).

Indeed, the group multiplication  $m: G \times G \rightarrow G$  is naturally lifted to a symplectic relation from  $\mathbf{Ph} G \times \mathbf{Ph} G$  to  $\mathbf{Ph} G$ , which obeys the axioms of symplectic groupoid (it is the *group  $S^*$ -algebra*). It defines the co-groupoid structure on  $T^*G$  (with the canonical symplectic structure). The left groupoid projection from  $T^*G$  to  $\mathfrak{g}$  consists in right translation to the group unit. The Poisson structure on  $\mathfrak{g}^*$  defined by this projection is dual to the Lie commutator on  $\mathfrak{g}$  coming from the right-invariant fields.

Now we shall describe the phase spaces of Poisson groups and motivate the first convention.

First let us recall that if  $(\mathfrak{g}, \partial r)$  is a coboundary Lie bialgebra (here  $r \in \bigwedge^2 \mathfrak{g}$  and  $\partial r: \mathfrak{g} \rightarrow \bigwedge^2 \mathfrak{g}$  is its coboundary) and  $G$  is the Lie group corresponding to  $\mathfrak{g}$  (convention 3), then the Poisson (group) structure on  $G$  corresponding to  $\partial r$  is given by  $G \ni g \mapsto rg - gr$ . In this case also  $g \mapsto rg + gr$  is a Poisson structure (not a Poisson group structure unless  $r = 0$ ).

Next, recall that for any Lie bialgebra  $(\mathfrak{g}, \delta)$ , its *Drinfeld double* is the coboundary Lie bialgebra  $(\mathfrak{g} \bowtie \mathfrak{g}^*, \partial r_D)$ , where  $r_D$  is the canonical element

$$r_D = \frac{1}{2} e_k \wedge e^k \in \bigwedge^2 (\mathfrak{g} \bowtie \mathfrak{g}^*) \quad (3)$$

( $e_k$  is a basis in  $\mathfrak{g}$  and  $e^k$  is the corresponding dual basis in  $\mathfrak{g}^*$ , both considered as elements of  $\mathfrak{g} \bowtie \mathfrak{g}^*$ ) and  $\mathfrak{g} \bowtie \mathfrak{g}^*$  is the Lie algebra of the Manin triple (of the Lie bialgebra). The Lie group  $M$  corresponding to  $\mathfrak{g} \bowtie \mathfrak{g}^*$  is a Poisson group (with the Poisson structure  $M \ni \xi \mapsto r_D \xi - \xi r_D$ ). Suppose that  $M$  *decomposes* on its two subgroups  $G, G^*$ , corresponding to Lie subalgebras  $\mathfrak{g}, \mathfrak{g}^*$ , i.e.  $G \cdot G^* = M$ ,  $G \cap G^* = \{e\}$ . In this case the Poisson group  $M$  is said to be the Drinfeld double of the Poisson group  $G$  (the Poisson group structure on  $G$  is defined by  $\delta$ ). It contains  $G$  as a Poisson subgroup (here the sign of (3) is important!). It is known that the

Poisson structure  $\pi_+(\xi) = r_D\xi + \xi r_D$  on  $M$  is nondegenerate (everywhere on  $M$ ) and  $M$  carries a natural structure of a (symplectic) (co-)groupoid over  $G$ , the left and right projections  $g$  and  $g'$  of an element  $\xi \in M$  on  $G$  being given by

$$\xi = g\gamma = \gamma'g', \quad \gamma, \gamma' \in G^*.$$

Moreover, the left projection of  $\pi_+$  on  $G$  is exactly the Poisson group structure on  $G$ , hence  $M = \mathbf{Ph} G$  (here also the sign in (3) is important!).

For the trivial cobracket  $\delta = 0$ ,  $M = T^*G$  and its symplectic structure turns out to be opposite of the standard one, due to the order of factors in  $r_D$ .

### 3 Generating functions

Let  $Q$  be a manifold (with zero Poisson structure). Any (smooth) function  $f: Q \rightarrow \mathbb{R}$  generates a Lagrangian submanifold  $L_f$  of  $\mathbf{Ph} Q \cong T^*Q$ , by the formula

$$L_f = \{df(x) : x \in Q\}. \quad (4)$$

We shall give another description of this construction (its generalization for the case of Poisson manifolds will be obvious). Recall that the Hamiltonian vector field  $X_\phi$  of a function  $\phi$  on a Poisson manifold  $(P, \pi)$  is defined by

$$X_\phi\psi = \{\phi, \psi\} = \pi(d\phi, d\psi),$$

where  $\psi$  is an arbitrary function on  $P$ . We denote by  $t \mapsto \exp(tX)$  the flow of a vector field  $X$ . Now observe that  $L_f$  given in (4) equals the image of the zero section of  $T^*Q$  by  $\exp(X_{f^{\text{left}}})$ :

$$L_f = \exp(X_{f^{\text{left}}})Q \quad (5)$$

(here we identify the zero section with  $Q$ ), where  $f^{\text{left}}$  is the pullback of  $f$  to  $\mathbf{Ph} Q$  by the left projection (which, of course, coincides with the right projection in the case of the cotangent bundle). Note that here the sign of the symplectic structure on the cotangent bundle is important: formula (5) is valid for the  $\mathbf{Ph} Q$  symplectic form (convention 1) equal  $dx^j \wedge dp_j$  (using the cotangent bundle coordinates implied by coordinates  $x^j$  on  $Q$ ).

**Definition.** Let  $Q$  be an integrable Poisson manifold and  $f: Q \rightarrow \mathbb{R}$  a complete function (i.e. a function whose Hamiltonian vector field is complete). Then  $L_f$  given by (5) is said to be the Lagrangian submanifold of  $\mathbf{Ph} Q$  *generated* by  $f$ .

Note that  $f^{\text{left}}$  is a complete function if and only if  $f$  is a complete function, hence the definition is correct.

**Lemma.** Formula (5) is equivalent to

$$L_f = (\mathbf{Ph} f)^T(\mathbb{R} \times \{1\}), \quad (6)$$

where  $\mathbf{Ph} f$  is the phase lift of  $f$  (i.e. the morphism of  $S^*$ -coalgebras [4] corresponding to  $f$ ),  $(\mathbf{Ph} f)^T$  denotes the transposed relation of  $\mathbf{Ph} f$  and

$$\mathbb{R} \times \{1\} = \{(e, t) \in \mathbf{Ph} \mathbb{R} : e \in \mathbb{R}, t = 1\}$$

is the Lagrangian submanifold of  $\mathbf{Ph} \mathbb{R}$  given by the ‘time’  $t = 1$ .

*Proof.* Let us calculate  $\mathbf{Ph} f$  by the method of characteristics [4]. We consider the coisotropic submanifold in  $\mathbf{Ph} Q \times \overline{\mathbf{Ph}} \mathbb{R}$  given by the single constraint  $f^{\text{left}}(\xi) = e$ . The characteristics are the integral curves of

$$X_{f^{\text{left}} - e} = X_{f^{\text{left}}} + X_e = X_{f^{\text{left}}} + \frac{\partial}{\partial t}.$$

An initial condition of the form  $(x; f(x), 0)$  evolves like  $t \mapsto (\exp(tX_{f^{\text{left}}})x; f(x), t)$ , hence the graph of  $\mathbf{Ph} f$  is

$$\{\exp(tX_{f^{\text{left}}})x; f(x), t) : x \in Q, t \in \mathbb{R}\}.$$

This implies (6). □

Since  $\mathbf{Ph} f$  can be calculated also using the right projections, from (6) it follows that in (5) we can replace  $f^{\text{left}}$  by  $f^{\text{right}}$  (the pullback of  $f$  to  $\mathbf{Ph} Q$  by the right projection) and the result will be the same.

Since  $X_{f^{\text{left}}}$  is right-invariant [3], we have the following property:

$$\exp(X_{f^{\text{left}}})\xi = L_f \cdot \xi \quad \text{for } \xi \in \mathbf{Ph} Q \quad (7)$$

(the dot on the right hand side is the groupoid multiplication). Similarly,

$$\exp(X_{f^{\text{right}}})\xi = \xi \cdot L_f \quad \text{for } \xi \in \mathbf{Ph} Q. \quad (8)$$

**Example 1.** Let  $\mathfrak{g}$  be a Lie algebra. Any  $X \in \mathfrak{g}$  is a linear function on  $Q = \mathfrak{g}^*$ . It generates a Lagrangian submanifold in  $\mathbf{Ph} \mathfrak{g}^* = T^*G$ , where  $G$  is the (connected and simply connected) Lie group corresponding to  $\mathfrak{g}$ . We have

$$f^{\text{left}}(\xi) = \langle X^{\text{right invariant}}, \xi \rangle = \langle X_{\text{left translation}}, \xi \rangle,$$

therefore  $f^{\text{left}}$  is just the canonical moment map of the left translations, evaluated on  $X$ . It follows that  $f^{\text{left}}$  generates left translations and  $L_f = \exp(X_{f^{\text{left}}})T_e^*G = T_{\exp X}^*G$  is the fiber of the cotangent bundle at  $g = \exp X$ .

**Example 2.** Let  $S$  be a simply connected symplectic manifold. Its phase space is the *pair* co-groupoid  $\mathbf{Ph} S = S \times \overline{S}$  with  $S \cong \{(x, x) : x \in S\}$  as the set of units. The left and right projections are

$$(x, y) \mapsto x \quad \text{and} \quad (x, y) \mapsto y,$$

respectively, hence for any function  $f$  on  $S$  we have

$$f^{\text{left}}(x, y) = f(x), \quad f^{\text{right}}(x, y) = f(y),$$

and

$$X_{f^{\text{left}}} = X_f \oplus 0, \quad X_{f^{\text{right}}} = 0 \oplus (-X_f).$$

Assuming that  $f$  is complete we have then

$$\exp(X_{f^{\text{left}}})(x, x) = (\exp(X_f)x, x), \quad \exp(X_{f^{\text{right}}})(x, x) = (x, \exp(-X_f)x),$$

and it follows that the Lagrangian submanifold generated by  $f$  in  $\mathbf{Ph}S$  is the graph of the symplectic transformation  $\exp X_f: S \rightarrow S$ .

The last example shows that in general it is very difficult to find the generating function of a given Lagrangian submanifold: one has to find a Hamiltonian flow which includes a given symplectic transformation (the generator needs not be unique; it is seen also in Example 1). Also the generation procedure is in general quite ineffective, because it requires to solve the equations of motion for some Hamiltonian. One can hope it becomes simpler for the Casimir functions of a given Poisson structure (functions which Poisson commute with all other functions, i.e. functions constant on symplectic leaves). Let us discuss shortly this case. Note that each Casimir function is complete. For such functions  $f^{\text{left}} = f^{\text{right}}$  and  $X_{f^{\text{left}}}$  is tangent both to left and right fibers. Let  $Q_0 \subset Q$  be the (open and dense) subset of points of the maximal rank of the Poisson structure. The corresponding part  $\mathbf{Ph}Q_0$  of  $\mathbf{Ph}Q$  carries a natural coisotropic foliation (level sets of the pullbacks of Casimir functions). The leaves of the corresponding characteristic isotropic foliation are spanned by the action of the Hamiltonian vector fields of the pullbacks of Casimir functions. Each such characteristic leaf must belong to the intersection of a left fiber and a right fiber. In fact the intersection coincides with the characteristic leaf, because the intersection of tangent spaces to left and right fibers are exactly the tangent spaces to the characteristic foliation. In particular, the intersection of the left and the right fiber over the same point  $x \in Q_0$  — so called *isotropy group* of  $x$  (cf. [3]) — is spanned by the action on  $x$  of flows of the Hamiltonian vector fields of pullbacks of Casimir functions. Let  $\xi, \eta$  be two elements of the isotropy group which can be written as

$$\xi = \exp(X_{f^{\text{left}}})x = L_f \cdot x, \quad \eta = \exp(X_{g^{\text{left}}})x = L_g \cdot x,$$

for some Casimir functions  $f, g$  (formula (7)). We have

$$\xi \cdot \eta = x \cdot L_f \cdot L_g \cdot x = x \cdot L_g \cdot L_f \cdot x = \eta \cdot \xi \quad (9)$$

because  $L_f \cdot L_g = L_g \cdot L_f$  for Casimir functions. The latter property follows from

$$L_f \cdot L_g \cdot \xi = \exp(X_{f^{\text{left}}})\exp(X_{g^{\text{right}}})\xi = \exp(X_{g^{\text{right}}})\exp(X_{f^{\text{left}}})\xi = L_g \cdot L_f \cdot \xi.$$

From (9) it follows that the isotropy group is abelian. Since  $X_{f^{\text{left}}}$  is right invariant, formula

$$(X_{f^{\text{left}}})|_x \mapsto \xi = \exp(X_{f^{\text{left}}})x$$

actually defines the exponential map for this group. We conclude that in order to calculate  $L_f$ , it is sufficient to calculate  $df(x)$  at each point  $x \in Q_0$ , and use the groupoid structure restricted to the isotropy subgroup of  $x$ :

$$L_f|_{Q_0} = \{\exp((df(x))^\sharp)x : x \in Q_0\}$$

(here  $\sharp$  denotes ‘raising of indices’ by the Poisson structure on  $\mathbf{Ph} Q$ ).

We remark that our result on the commutativity of the ‘minimal’ isotropy subgroups (isotropy subgroups of maximal symplectic leaves) in the case of the ‘group  $S^*$ -algebra’ (cf. Section 2) means that isotropy subgroups of elements belonging to maximal coadjoint orbits are abelian (cf. [6]). In the case of ‘Poisson group  $S^*$ -algebra’ (i.e. the symplectic groupoid  $G \cdot G^*$  over  $G^*$ ), it means the same, with coadjoint action replaced by the dressing one.

**Example 3.** Let  $V$  be a vector space with a constant Poisson structure  $r$ , i.e. the Poisson brackets of linear coordinates  $x^j$  are

$$\{x^j, x^k\} = r^{jk} = -r^{kj},$$

where  $r^{jk}$  are some constants (the coefficients of  $r$  in the corresponding basis). The phase space  $\mathbf{Ph} V$  can be realized [7] in  $T^*V = V \oplus V^*$  with the canonical structure

$$\{q^j, q^k\} = 0, \quad \{q^j, p_k\} = \delta_k^j, \quad \{p_j, p_k\} = 0$$

and the left projection equal

$$x = (q, p)_L = q + \frac{1}{2}r(p), \quad [r(p)]^j := p_k r^{kj}.$$

For any *linear* function  $f(x) = a_j x^j$ , we can find the flow of  $f^{\text{left}}(q, p) = a_j(q^j + \frac{1}{2}p_k r^{kj})$ . We have

$$\dot{q}^k = \{f^{\text{left}}, q^k\} = \frac{1}{2}a_j r^{jk}, \quad \dot{p}_j = a_j,$$

i.e.  $x^k(t) = x^k(0) + \frac{1}{2}a_j r^{jk}t$ ,  $p_j = p_j(0) + a_j t$ , hence

$$L_f = \{(q, p) : p = a\}$$

is the same ‘horizontal’ Lagrangian plane as in the usual case.

If  $f$  is *quadratic*, then  $f^{\text{left}}$  is also quadratic and  $\exp(X_{f^{\text{left}}})$  is a linear symplectic transformation, hence  $L_f$  is a Lagrangian (linear) subspace of  $V \oplus V^*$ , as in the usual case.

## 4 Particle on the Poisson $SU(N)$

An explicit calculation of the relation between the Lagrangian and the Hamiltonian may be very difficult in the ‘twisted’ case, due to the nontrivial generation procedure. In this section we analyze this problem a little bit for an analogue of the free motion, when the Poisson configuration manifold  $Q$  is the standard Poisson  $SU(N)$  group. The description of  $\mathbf{Ph} Q$  is given in [8]: it is  $SL(N, \mathbb{C})$  — the Manin group (with the  $\pi_+$  Poisson structure), which contains both  $G = SU(N)$  and its (Poisson) dual  $G^* = SB(N)$ . We can ask two questions:

1. Which Lagrangian corresponds to the Hamiltonian

$$\mathcal{H}(g) := \frac{1}{2} \text{tr } g^\dagger g, \quad g \in SL(N, \mathbb{C}), \quad (10)$$

introduced in [8] and describing a natural analogue of the free motion on  $SU(N)$ ?

2. Which Hamiltonian corresponds to the ‘non-deformed free’ Lagrangian

$$\mathcal{L}(v) := \frac{1}{2} v^2, \quad v \in TQ, \quad (11)$$

where the square of  $v \in TQ = TG$  is taken with respect of a bi-invariant metric on  $Q = G$ ?

Unfortunately, we are still premature to answer these questions. In the sequel we show how far we are able to reduce the problem.

The dynamical equations implied by (10) were obtained in [8] and look as follows

$$\dot{g} = i\varepsilon[gg^\dagger g - \frac{1}{N}(\text{tr } g^\dagger g)g]. \quad (12)$$

If we decompose  $g = u\gamma$  with  $u \in G$ ,  $\gamma \in G^*$ , we get

$$\dot{\gamma} = 0, \quad u^{-1}\dot{u} = F(\gamma) := i\varepsilon[\gamma\gamma^\dagger - \frac{1}{N}(\text{tr } \gamma\gamma^\dagger)]. \quad (13)$$

It was shown in [8] that  $F$  is bijective, hence the dynamics (12) is a section of the left projection

$$TSL(N, \mathbb{C}) \ni \dot{g} = \dot{u}\gamma + u\dot{\gamma} \mapsto \dot{u} \in TG,$$

namely

$$TG \ni \dot{u} \mapsto \dot{g} = \dot{u}F^{-1}(u^{-1}\dot{u}) \in TSL(N, \mathbb{C}).$$

Similarly, decomposing  $g$  in the opposite order,  $g = \gamma u$ , we get

$$\dot{\gamma} = 0, \quad \dot{u}u^{-1} = E(\gamma) := i\varepsilon[\gamma^\dagger\gamma - \frac{1}{N}(\text{tr } \gamma^\dagger\gamma)]. \quad (14)$$

Since  $E$  is also bijective, the dynamics (12) is also a section of the right projection

$$TSL(N, \mathbb{C}) \ni \dot{g} = \gamma\dot{u} + \dot{\gamma}u \mapsto \dot{u} \in TG,$$

namely

$$TG \ni \dot{u} \mapsto \dot{g} = E^{-1}(\dot{u}u^{-1})\dot{u} \in TSL(N, \mathbb{C}). \quad (15)$$

In order to find its generating function on  $TQ$  — the Lagrangian — we have to try to calculate the Lagrangian submanifolds generated by some ‘natural’ (class of) Lagrangian(s) and see whether this fits the original dynamics. Thus we are in fact led to the Question 2 above.

In order to find the Lagrangian submanifold generated by a function on  $TG$ , we should have a formula for the left projection from  $T\mathbf{Ph}Q$  to  $TQ$  and an effective



control of the symplectic structure of  $T\mathbf{Ph}Q$ . For this latter purpose, it is more convenient to work in the canonical symplectic structure of  $T^*\mathbf{Ph}Q$ . We can use it passing from  $T\mathbf{Ph}Q$  to  $T^*\mathbf{Ph}Q$  with the help of  $\beta$  (formula (2)). Fortunately, the isomorphism  $\beta$  not only preserves the symplectic structure, but also transforms the tangent groupoid into cotangent groupoid of  $\mathbf{Ph}Q$ . Indeed, as can be easily seen, for any symplectic relation  $\rho: X \rightarrow Y$  (from a symplectic manifold  $X$  to a symplectic manifold  $Y$ ),  $T\rho$  is transformed into  $\mathbf{Ph}\rho$  by  $\beta$ . We just apply this fact to the case of the groupoid multiplication relation in  $\mathbf{Ph}Q$ . It follows that the left projection in  $T\mathbf{Ph}Q$  over  $TQ$  is just the left projection in  $T^*\mathbf{Ph}Q$  over  $(TQ)^\circ$  (the conormal bundle to  $TQ$  in  $T^*\mathbf{Ph}Q$ ), composed with the inverse of  $\beta$ .

The left projection in  $T^*\mathbf{Ph}Q$  over  $(TQ)^\circ$  for  $Q$  being a double Lie group has been calculated in [4] (formula (25)). It is given by

$$\xi_L = [P(\xi(u\gamma)^{-1})]u \quad \text{for } \xi \in T_{u\gamma}^*(G \cdot G^*), \quad (16)$$

where  $P$  is the projection on  $\mathfrak{g}^\circ$  parallel to  $(\mathfrak{g}^*)^\circ$  (here  $u \in G, \gamma \in G^*$ ). Let  $s$  be the canonical symmetric map from  $(\mathfrak{g} \ltimes \mathfrak{g}^*)^*$  to  $\mathfrak{g} \ltimes \mathfrak{g}^*$ . We have

$$[(su)\xi_L]u^{-1} = s(\xi_L u^{-1}) = s(P(\xi g^{-1})) = \xi g^{-1}|_{\mathfrak{g}^*} \in \mathfrak{g} \quad \text{for } g = u\gamma \in G \cdot G^*.$$

Using the formula for  $\sharp = \beta^{-1}$ ,

$$\sharp = \frac{1}{2}(Rg + gR)(gs) = \frac{1}{2}(Rg + gR)(sg)$$

(which follows from  $\pi_+ = r_D g + g r_D$ ), where  $R$  is the reflection in  $\mathfrak{g}^*$  parallel to  $\mathfrak{g}$ , we obtain

$$\sharp \xi_L = \frac{1}{2}(Ru + uR)(su)\xi_L = -(su)\xi_L = -[\xi g^{-1}|_{\mathfrak{g}^*}]u.$$

If  $\mathcal{L}: TG \rightarrow \mathbb{R}$  is right-invariant, then

$$\mathcal{L}(\sharp \xi_L) = \mathcal{L}(-(\xi g^{-1})|_{\mathfrak{g}^*})$$

is a right-invariant function on  $T^*(G \cdot G^*)$ . For instance, if  $\mathcal{L}$  is given by a right-invariant metric on  $G$ , such that for  $v \in \mathfrak{g}$  we have

$$\mathcal{L}(v) = \frac{1}{2} \langle l, v \otimes v \rangle,$$

where  $l$  is a symmetric element of  $\mathfrak{g}^* \otimes \mathfrak{g}^*$ , then

$$\mathcal{L}(\sharp \xi_L) = \frac{1}{2} \langle l, \xi g^{-1}|_{\mathfrak{g}^*} \otimes \xi g^{-1}|_{\mathfrak{g}^*} \rangle = \frac{1}{2} \langle l, \xi g^{-1} \otimes \xi g^{-1} \rangle = \frac{1}{2} \langle l g, \xi \otimes \xi \rangle,$$

so we are interested in the flow of the right-invariant contravariant metric on  $G \cdot G^*$  which at the unit is tangent to  $\mathfrak{g}^*$  (and equal  $l$ ). Of course, in such a situation, the left translation of  $\xi$  to the group unit is constant:

$$g^{-1}\xi = \xi_0 \in (\mathfrak{g} \ltimes \mathfrak{g}^*)^*.$$

Since we are considering the flow of a function which is constant on the fibers of the left projection, we know that the right projection, given by the formula analogous to (16),

$$\xi_R = uP((\gamma u)^{-1}\xi) \quad \text{for } \xi \in T_{\gamma u}^*(G \cdot G^*), \quad (17)$$

is preserved. It follows that  $u$  is constant and

$$\xi = g\xi_0 = \gamma(t)u\xi_0.$$

We are interested in the initial conditions  $\gamma(0) = e$ ,  $\xi_0 \in \mathfrak{g}^\circ$ , hence

$$\xi = \gamma(t)\eta_0 u \quad (18)$$

for some  $\eta_0 \in \mathfrak{g}^\circ$ . Due to the right invariance, it is sufficient to solve the flow for the initial condition being just  $\eta_0 \in \mathfrak{g}^\circ \cong \mathfrak{g} \cong (\mathfrak{g}^*)^*$  (i.e.  $u = e$ ), hence  $\xi = \gamma(t)\eta_0$ . Now it is easy to see that the problem is reduced to solving the flow on

$$T^*G^* \cong G^* \cdot \mathfrak{g}^\circ \subset T^*(G \cdot G^*),$$

generated by the right-invariant contravariant metric on  $G^*$ , defined by  $l$ , for the initial condition  $\eta_0 \in \mathfrak{g}$ . Note that the flow of a right-invariant positive contravariant metric is always complete on the cotangent bundle, because such a Hamiltonian is a pullback (by the left projection) of a function on the dual of the Lie algebra which has compact level sets (hence the function is complete).

We conclude that the whole information about the Lagrangian submanifold generated by (11) is contained in the ‘one-fiber’ Legendre transformation

$$\mathfrak{g} \cong \mathfrak{g}^\circ \ni \eta_0 \mapsto \Phi(\eta_0) \in G^*,$$

where  $\Phi(\eta_0) = \gamma(1)$  is the ‘time equal 1’ point of the geodesic  $t \mapsto \gamma(t)$  which starts at  $t = 0$  with velocity corresponding to  $\eta_0$ . The Lagrangian submanifold generated by  $\mathcal{L}$  is described by

$$\xi = \Phi(\eta_0)\eta_0 u, \quad \eta_0 \in \mathfrak{g}^\circ \cong \mathfrak{g}, \quad u \in G. \quad (19)$$

In order to compare it with (15), let us note that  $\dot{g} \in g\mathfrak{g} \cap \mathfrak{g}g$  (because both left and right projections of  $\dot{g}$  on  $G^*$  are zero), therefore

$$\frac{1}{2}(Rg + gR)\dot{g} = -\dot{g}.$$

It follows that

$$\begin{aligned} \beta(\dot{g}) &= -(sg)(\dot{g}) = -(sg)[E^{-1}(\dot{u}u^{-1})\dot{u}] \\ &= -(sg)[E^{-1}(v)vu] = -E^{-1}(v)s(v)u, \end{aligned}$$

where  $v := \dot{u}u^{-1} \in \mathfrak{g}$ , hence we see that  $\mathcal{L}$  and  $\mathcal{H}$  generate the same dynamics when

$$-E^{-1} = \Phi. \quad (20)$$

Thus we have reduced the problem of comparing the Hamiltonian and the Lagrangian dynamics to the equality of the above type. Some straightforward calculations for  $SU(2)$  show that the equality does not hold for  $\mathcal{H}$  and  $\mathcal{L}$  as given by (10) and (11), respectively. The above reduction of the problem to the form (20) is valid however for  $\mathcal{H}$  coming from any Casimir on  $G^*$  and  $\mathcal{L}$  coming from any invariant on  $\mathfrak{g}$ . Therefore there is still room for finding appropriate compatible pairs  $(\mathcal{H}, \mathcal{L})$ .

## References

- [1] W. M. Tulczyjew, *Hamiltonian systems, Lagrangian systems and the Legendre transformation*, Symposia Matematica **14** (1974), 247–258.  
W. M. Tulczyjew, *The Legendre transformation*, Ann. Inst. H. Poincaré **27** (1977), 101–114.
- [2] W. M. Tulczyjew, Geometric formulations of physical theories, Lecture Notes ed. by Bibliopolis, Napoli 1989.
- [3] A. Coste, P. Dazord and A. Weinstein, *Groupeïdes symplectiques*, Publications du Département de Mathématiques, Université Claude Bernard Lyon I (1987).
- [4] S. Zakrzewski, *Quantum and classical pseudogroups. Part I and II*, Comm. Math. Phys. **134** (1990), 347–395.
- [5] J. Grabowski and P. Urbański, *Tangent lifts of Poisson and related structures*, J. Phys. A: Math. Gen. **28** (1995) 6743–6777.
- [6] M. Duflo and M. Vergne, C. R. Acad. Sci. Paris, Ser. A, **268** (1969), 583–A585.  
J. Dixmier, Algèbres enveloppantes, Paris 1974, Gauthier-Villars.  
V. Guillemin and S. Sternberg, Geometric asymptotics, AMS Providence, 1977.
- [7] S. Zakrzewski, *Geometric quantization of Poisson groups — diagonal and soft deformations*, Proceedings of the Taniguchi Symposium *Symplectic geometry and quantization problems*, Sanda (1993), Y. Maeda, H. Omori and A. Weinstein (Eds.), Contemporary Mathematics **179**, 1994, 271–285.
- [8] S. Zakrzewski, *Free motion on the Poisson  $SU(N)$  group*, Warsaw preprint 1996.